International Journal of Theoretical Physics, Vol. 33, No. 11, 1994

Sums and Products of Interval Algebras

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Received August 17, 1994

An *interval algebra* is an interval from zero to some positive element in a partially ordered Abelian group, which, under the restriction of the group operation to the interval, is a partial algebra. In this paper we study interval algebras from a categorical point of view, and show that Cartesian products and horizontal sums are effective as categorical products and coproducts, respectively. We show that the category of interval algebras admits a tensor product, and introduce a new class of interval algebras, which are in fact orthoalgebras, called χ-algebras.

1. INTRODUCTION

By an interval algebra, we mean an interval $G^+[0, u] = \{g \in G | 0 \leq g \leq u\}$ in a partially ordered Abelian group G, organized into a partial algebra under the partially defined binary operation $\oplus$ obtained by restriction to $G^+[0, u]$, of the group operation $+$ on G. The prototype for such an algebra is $\mathcal{V}^+[\mathbb{0}, \mathbb{1}]$, where $\mathcal{V}$ is the additive group of self-adjoint operators on a Hilbert space. We recall that elements of $\mathcal{V}^+[\mathbb{0}, \mathbb{1}]$ are called *effects* and that effect-valued measures play an important role in the stochastic approach to quantum mechanics (Ali, 1985; Beltrametti and Cassinelli, 1981; Prugovecki, 1986; Schroeck and Foulis, 1990).

In what follows, we assume the reader is familiar with the material in Bennett and Foulis (n.d.) and Foulis and Bennett (1994), although, for convenience, we shall reproduce some of the basic definitions and results. Effect algebras are mathematically equivalent to the weak orthoalgebras of Giuntini and Greuling (1989) and to the D-posets of Kôpka and Chovanec

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0020-7748/94/1100-2119$07.00/0

(n.d.), Kôpka and Pták (1993) and they are closely related to BZ-posets (Cattaneo and Nistico, 1989). Effect algebras in general, and interval algebras in particular, can be regarded as (possibly) unsharp quantum logics (Della Chiara and Giuntini, 1989; Giuntini and Greuling, 1989).

Our main purpose in this paper is to show that the Cartesian product and horizontal sum of interval algebras are again interval algebras, and that there is a tensor product in the category of interval algebras. For a physical interpretation of sums, Cartesian products, and tensor products of quantum logics, see Foulis (1989). We also present a number of illustrative examples of interval algebras and introduce a new class of orthoalgebras called χ-algebras.

2. EFFECT ALGEBRAS

In Foulis and Bennett (1994) an *effect algebra* is defined to be an algebraic system $(A, 0, u, \oplus)$ consisting of a set A, two special elements $0, u \in A$ called the *zero* and the *unit*, and a partially defined binary operation $\oplus$ on A that satisfies the following conditions for all $p, q, r \in A$:

(i) [*Commutative Law*] If $p \oplus q$ is defined, then $q \oplus p$ is defined and $p \oplus q = q \oplus p$.
(ii) [*Associative Law*] If $q \oplus r$ is defined and $p \oplus (q \oplus r)$ is defined, then $p \oplus q$ is defined, $(p \oplus q) \oplus r$ is defined, and $p \oplus (q \oplus r) = (p \oplus q) \oplus r$.
(iii) [*Orthosupplement Law*] For every $p \in A$ there exists a unique $q \in A$ such that $p \oplus q$ is defined and $p \oplus q = u$.
(iv) [*Zero–Unit Law*] If $u \oplus p$ is defined, then $p = 0$.

An effect algebra A is partially ordered by the relation $\leq$ defined by $p \leq q$ iff there is an $r \in A$ with $p \oplus r = q$. The order structure $(A, \leq)$ of the effect algebra A is derived from its algebraic structure $(A, 0, u, \oplus)$, but not vice versa. There are posets (partially ordered sets) that can be organized into effect algebras in more than one way, and there are posets (even finite distributive lattices) that cannot be organized into effect algebras at all. If A is totally ordered by $\leq$, it is called a *scale algebra*. If $(A, \leq)$ is a lattice, we say that A is *lattice ordered*.

Let A be an effect algebra and let $p \in A$. We define $0p = 0$ and $1p = p$. More generally, if n is a positive integer and $(n-1)p$ is defined, we say that np is defined iff $(n-1)p \oplus p$ is defined, in which case $np := (n-1) \oplus p$. (We use the notation $:=$ to mean *equals by definition*.) The element p is said to be *isotropic* iff $p \neq 0$ and $2p = p \oplus p$ is defined. If there is a largest positive integer n for which np is defined, then n is called the *isotropic index* of p. If np is defined for all positive integers n, we say that p has *infinite* isotropic index. An *orthoalgebra* (Bennett and Foulis, 1993; Foulis *et al.*, 1992; Nevara and Pták, 1993) may be characterized as an effect algebra with no isotropic elements. Therefore, Boolean algebras, orthomodular lattices (Beran, 1984; Kalmbach, 1983) and orthomodular posets (Kalmbach) are all special cases of effect algebras.

Let A, B, and C be effect algebras with units u, v, and w, respectively. A mapping $\phi: A \to B$ is *additive* iff, whenever $p, q \in A$ and $p \oplus q$ is defined in A, $\phi(p) \oplus \phi(q)$ is defined in B and $\phi(p \oplus q) = \phi(p) \oplus \phi(q)$. An additive mapping $\phi: A \to B$ is called a *morphism* iff $\phi(u) = v$. A mapping $\theta: A \times B \to C$ is a *bimorphism* iff, for all $a \in A$ and $b \in B$, $\theta(\cdot, b): A \to C$ and $\theta(a, \cdot): B \to C$ are additive mappings and $\theta(u, v) = w$.

If K is an Abelian group, a mapping $\phi: A \to K$ is a *K-valued measure* iff, whenever $p, q \in A$ and $p \oplus q$ is defined in A, $\phi(p \oplus q) = \phi(p) + \phi(q)$. A mapping $\theta: A \times B \to K$ is a *K-valued bimeasure* iff, for all $a \in A$ and $b \in B$, $\theta(\cdot, b): A \to K$ and $\theta(a, \cdot): B \to K$ are K-valued measures.

A *sub-effect algebra* of an effect algebra A with unit u is a subset S of A such that $0, u \in S$, $p \in S \Rightarrow \exists r \in S$ with $p \oplus r = u$, and $p, q \in S$ with $p \oplus q = s \Rightarrow s \in S$. Such a sub-effect algebra S is an effect algebra in its own right under the restriction to S of $\oplus$ on A.

3. INTERVAL ALGEBRAS

If G is an additively-written partially ordered Abelian group, we denote the *positive cone* in G by $G^+ := \{g \in G | 0 \le g\}$ and, if $0 \ne u \in G^+$, we define the *interval* $G^+[0, u] := \{g \in G | 0 \le g \le u\}$. The interval $G^+[0, u]$ can be organized into an effect algebra $(G^+[0, u], 0, u, \oplus)$ by defining $p \oplus q$ iff $p + q \le u$, in which case $p \oplus q := p + q$. An effect algebra of the form $G^+[0, u]$, or isomorphic to such an effect algebra, is called an *interval effect algebra* or simply an *interval algebra* for short (Bennett and Foulis, n.d.). We use the notation $\mathbb{Z}^+$ and $\mathbb{R}^+$ for the standard positive cones in the additive groups $\mathbb{Z}$ of integers and $\mathbb{R}$ of real numbers ordered in the usual way.

The following three theorems are proved in Bennett and Foulis (n.d.).

Theorem 3.1. A sub-effect algebra of an interval algebra is again an interval algebra.

Theorem 3.2. If A is an interval algebra, there exists a partially ordered Abelian group G and an element $0 \ne u \in G^+$ such that:

(i) $A = G^+[0, u]$ is an interval algebra.
(ii) $G = G^+ - G^+$, i.e., G^+ is a generating cone in G.
(iii) Every element $g \in G^+$ has the form $g = a_1 + a_2 + \cdots + a_n$ for a finite sequence $a_1, a_2, \ldots, a_n \in A$.

(iv) If K is an Abelian group, then every K-valued measure $\phi: A \to K$ can be extended to a group homomorphism $\phi^*: G \to K$.

The partially ordered Abelian group G in Theorem 3.2, which is unique up to an isomorphism, is called the *universal group with unit u* for A. Theorem 3.2 will be our main tool for the study of sums and products of interval algebras.

Theorem 3.3. Every scale algebra is an interval algebra in a totally ordered Abelian group. Furthermore, if G is a totally ordered Abelian group, $0 \neq u \in G^+$, and every element in G^+ is a sum of a finite sequence of elements in $A := G^+[0, u]$, then G is the universal group for the scale algebra A.

The universal group provides a natural basis for the following notation of a *multiple* of an interval algebra.

Definition 3.4. If G is the universal group with unit u for the interval algebra $A = G^+[0, u]$, and if n is a positive integer, we define $nA := G^+[0, nu]$.

Lemma 3.5. Let G be the universal group with unit u for the interval algebra $A = G^+[0, u]$ and let n be a positive integer. Then G is the universal group with unit nu for nA.

Proof. Obviously, conditions (i)–(iii) of Theorem 3.2 are satisfied. To prove (iv), suppose K is an Abelian group and $\phi: G^+[0, nu] \to K$ is a K-valued measure. Let $\psi: A \to K$ be the restriction of ϕ to $A = G^+[0, u]$. Then ψ is a K-valued measure on A, so there is a group homomorphism $\psi^*: G \to K$ that extends ψ. If $g \in nA$, then $g \in G^+$, and it follows from part (iii) of Theorem 3.2 that there is a finite sequence $a_1, a_2, \ldots, a_n \in A$ such that $g = \sum_i a_i$. Since $0 \le g \le nu$, it follows that $g = a_1 \oplus' a_2 \oplus' \cdots \oplus' a_n$, where $\oplus'$ denotes orthogonal summation in nA. Therefore, $\phi(g) = \sum_i \phi(a_i)$ and $\psi^*(g) = \sum_i \psi^*(a_i) = \sum_i \phi(a_i) = \phi(g)$, so ψ^* is an extension of ϕ to G. ■

4. EXAMPLES

In this section, we give several examples of interval algebras. These examples will help to fix ideas and some of them are useful for constructing counterexamples.

Example 4.1. The simplest possible interval algebra is $\mathbf{2} := \mathbb{Z}^+[0, 1] = \{0, 1\}$. Note that $\mathbf{2}$ is the only orthoalgebra that is also a scale algebra and, as a poset, $\mathbf{2}$ is the two-element Boolean algebra.

Example 4.2. If n is a positive integer, we define the *n-chain* $C_n := n\mathbf{2} = \mathbb{Z}^+[0, n]$. Evidently, C_n is a scale algebra and, by Lemma 3.5, $\mathbb{Z}$ with the usual order is its universal group. Every finite scale algebra with $n + 1$ elements is isomorphic to the n-chain C_n.

If r is a positive integer, we define $\mathbb{Z}^r$ to be the r-fold Cartesian product $\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ of the additive Abelian group $\mathbb{Z}$ with itself. The *standard positive cone* $(\mathbb{Z}^+)^r$ in $\mathbb{Z}^r$ is understood to be the r-fold Cartesian product $\mathbb{Z}^+ \times \mathbb{Z}^+ \times \cdots \times \mathbb{Z}^+$.

Example 4.3. If $n_1, n_2, \ldots, n_r$ is a finite sequence of positive integers, we define the *rectangular trellis*

$$\mathrm{RT}(n_1, n_2, \ldots, n_r) := (\mathbb{Z}^+)^r[(0, 0, \ldots, 0), (n_1, n_2, \ldots, n_r)]$$

As a poset, $\mathrm{RT}(n_1, n_2, \ldots, n_r)$ forms a finite distributive lattice. We define the interval algebra $\mathbf{2}^r := \mathrm{RT}(n_1, n_2, \ldots, n_r)$ for $n_1 = n_2 = \cdots = n_r = 1$. As a poset, $\mathbf{2}^r$ is isomorphic to the Boolean algebra with $\mathbf{2}^r$ elements.

If X is a set, we denote by $\mathbb{Z}^X$ the set of all functions $f: X \to \mathbb{Z}$ organized into an additive Abelian group under pointwise operations. The *standard positive cone* in $\mathbb{Z}^X$ is understood to be the subset $(\mathbb{Z}^+)^X$ consisting of all functions $f \in \mathbb{Z}^X$ such that $f(x) \in \mathbb{Z}^+$ for all $x \in X$.

Example 4.4. Let X be a Stone space, i.e., a compact, Hausdorff, totally disconnected topological space. Let G be the subgroup of $\mathbb{Z}^X$ consisting of all functions $f: X \to \mathbb{Z}$ that are continuous when $\mathbb{Z}$ is given the discrete topology. Partially order G by the positive cone $G^+ := G \cap (\mathbb{Z}^+)^X$ and let $u \in G$ be the constant function $u(x) = 1$ for all $x \in X$. Then G is the universal group for the interval algebra $G^+[0, u]$, and the Boolean algebra of all compact open subsets M of X is isomorphic as a sublattice to $G^+[0, u]$ under the mapping $M \mapsto \chi_M$ that carries M into the characteristic set function χ_M of M. Thus, by Stone's theorem (Stone, 1936), every Boolean algebra can be organized into an interval algebra.

Example 4.5. The *standard scale algebra* $\mathbb{R}^+[0, 1]$ has $\mathbb{R}$, ordered in the usual way, as its universal group. A scale algebra is isomorphic to a sub-effect algebra of $\mathbb{R}^+[0, 1]$ iff it has no nonzero elements of infinite isotropic index.

If G and H are partially ordered Abelian groups, the group $P := G \times H$, partially ordered by the positive cone

$$P^+ := \{(g, h) \in G \times H \mid 0 \neq g \in G^+ \text{ or } (g = 0 \text{ and } h \in H^+)\}$$

is called the *lexicographic product* of G and H. If G and H are totally ordered, so is their lexicographic product.

Example 4.6. Let $\mathbb{Z}$ be ordered by the standard positive cone $\mathbb{Z}^+$ and let $P := \mathbb{Z} \times \mathbb{Z}$ be the lexicographic product. Then, in the scale algebra

$A := P^+[(0,0),(1,0)]$, every element of the form $(0,k)$, $k \in \mathbb{Z}^+$, has infinite isotropic index. By Theorem 3.3, P is the universal group for A.

Example 4.7. Let $\mathbb{Z}$ be ordered by the standard positive cone $\mathbb{Z}^+$, let $0 \neq m, n \in \mathbb{Z}^+$, let $\mathbb{Z}_m$ denote the additive group of integers modulo m partially ordered by the trivial cone $(\mathbb{Z}_m)^+ = \{0\}$, and let $P := \mathbb{Z} \times \mathbb{Z}_m$ be the lexicographic product. We define the *polychain of height n* and *width m* by $C_{n,m} := P^+[(0,0),(n, m-1)]$. For $m = 2$, we define the *diamond* by $D := P^+[(0,0),(2,0)]$. The diamond D and $C_{2,2}$ are isomorphic *as posets*, but not as effect algebras, whereas $C_{2,2}$ and $\mathbf{2}^2$ are isomorphic as effect algebras.

The polychains $C_{n,m}$ are mainly useful for constructing counterexamples. As a poset, the elements of $C_{n,m}$ are arranged in $n+1$ "levels" with 0 alone in the bottom level, $(n, m-1)$ alone in the top level, and m elements in each of the $n-1$ additional levels. Distinct elements in each level are incomparable, whereas every element in each level is less than every element in the next higher level. For $n \geq 3$, it can be shown that the lexicographically ordered group $\mathbb{Z} \times \mathbb{Z}_m$ in Example 4.7 is the universal group for $C_{n,m}$ and $\mathbb{Z} \times \mathbb{Z}_2$ is the universal group for the diamond D; however, $\mathbb{Z} \times \mathbb{Z}_2$ is not the universal group for $C_{2,2}$.

Example 4.8. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{V}$ be the partially ordered real Banach space of all self-adjoint operators on $\mathcal{H}$. The interval algebra $\mathcal{E}(\mathcal{H}) := \mathcal{V}^+[\mathbb{0}, \mathbb{1}]$, called the *standard effect algebra* on $\mathcal{H}$, supplies much of the motivation for the study of effect algebras. If $0 \neq T \in \mathcal{E}(\mathcal{H})$, the isotropic index of T is the largest positive integer n such that the spectrum of T is contained in the interval $[0, 1/n]$. The sub-effect algebra $\mathbb{P}(\mathcal{H})$ of $\mathcal{E}(\mathcal{H})$ consisting of all idempotents in $\mathcal{E}(\mathcal{H})$ is the *standard quantum logic* (Piron, 1976). By Theorem 3.1, $\mathbb{P}(\mathcal{H})$ is an interval algebra.

5. PROBABILITY MEASURES

If A is an effect algebra, a morphism $\omega: A \to \mathbb{R}^+[0,1]$ is called a *probability measure* on A. We denote by $\Omega(A)$ the set of all probability measures on A. Evidently, $\Omega(A)$ is a convex subset of the vector space $\mathbb{R}^A$ of all real-valued functions on A. The set of all extreme points of a convex set A is denoted by $\partial_e \Delta$. In quantum logic (Beltrametti and Cassinelli, 1981; Greechie and Gudder, 1975; Gudder, 1988; Pták and Pulmannová, 1991), elements of $\Omega(A)$ are called *states* and elements of $\partial_e \Omega(A)$ are called *pure states*. We say that the effect algebra A admits an *order-determining* set of probability measures iff, whenever $p, q \in A$ and $\omega(p) \leq \omega(q)$ for all $\omega \in \Omega(A)$, it follows that $p \leq q$ in A.

The following two theorems are proved in Bennett and Foulis (n.d.).

Theorem 5.1. If an effect algebra admits an order-determining set of probability measures, then it is an interval algebra.

Theorem 5.2. Every interval algebra admits at least one probability measure.

It can be shown that a scale algebra A admits exactly one probability measure ω and that A is a sub-effect algebra of the standard scale algebra $\mathbb{R}^+[0, 1]$ iff $\{\omega\}$ is an order-determining set of probability measures on A. In Example 4.6, the unique probability measure ω satisfies $\omega(x, y) = 0$ if $y > 0$ and $\omega(x, y) = 1$ if $y \leq 0$, so it fails to be order determining. If $n \geq 3$, the polychain $C_{n,m}$ in Example 4.7 admits exactly one probability measure ω given by $\omega(x, y) = x/n$; furthermore, $\{\omega\}$ is an order-determining set of probability measures on $C_{n,m}$.

Example 5.3. Although the diamond D and the interval algebra $\mathbf{2}^2$ are isomorphic as posets, $\Omega(D)$ consists of a single probability measure, whereas $\Omega(\mathbf{2}^2)$ is affine-isomorphic to the unit interval $[0, 1] \subseteq \mathbb{R}$.

6. PRODUCTS AND SUMS OF EFFECT ALGEBRAS

Let A and B be effect algebras with units u and v, respectively. The Cartesian product $A \times B$ can be organized into an effect algebra with unit (u, v) in such a way that $(a_1, b_1) \oplus (a_2, b_2)$ is defined in $A \times B$ iff $a_1 \oplus a_2$ is defined in A and $b_1 \oplus b_2$ is defined in B, in which case, $(a_1, b_1) \oplus (a_2, b_2) := (a_1 \oplus a_2, b_1 \oplus b_2)$. An n-fold Cartesian product $A_1 \times A_2 \times \cdots \times A_n$ is defined in the obvious way.

Example 6.1. The rectangular trellis $\mathrm{RT}(n_1, n_2, \ldots, n_r)$ in Example 4.3 is the Cartesian product $C_{n_1} \times C_{n_2} \times \cdots \times C_{n_r}$ of the chains $C_{n_1}, C_{n_2}, \ldots, C_{n_r}$ in Example 4.2.

If r is a positive integer, we understand that A^r is the effect algebra obtained by forming the r-fold Cartesian product of A with itself. In particular, as a poset, the effect algebra $\mathbf{2}^r$ is the finite Boolean algebra with 2^r elements, However, for $r \geq 2$, the Boolean algebra with 2^r elements, regarded simply as a poset, can always be organized into an effect algebra in more than one way.

To form the horizontal sum $A \dotplus B$ of A and B, we begin by relabeling the elements of A and B, if necessary, so that $A \cap B = \{0, w\}$, where $w = u = v$. The *horizontal sum* is then defined to be $A \dotplus B := A \cup B$, organized into an effect algebra in such a way that, for $x, y \in A \dotplus B$, $x \oplus y$ is defined iff $x, y \in A$ or $x, y \in B$, in which case $x \oplus y$ is defined as in A or B, respectively.

Example 6.2. The diamond D in Example 4.7 is isomorphic to the horizontal sum $C_2 \dotplus C_2$ of two 2-chains.

An n-fold horizontal sum $A_1 \dotplus A_2 \dotplus \cdots \dotplus A_n$ is defined in the obvious way.

Example 6.3. If n is a positive integer, the effect algebra $MO(n)$ is defined to be the horizontal sum $\mathbf{2}^2 \dotplus \mathbf{2}^2 \dotplus \cdots \dotplus \mathbf{2}^2$ of n copies of the interval algebra $\mathbf{2}^2$. As a poset, $MO(n)$ forms a finite modular orthocomplemented lattice (Kalmbach, 1983, p. 29).

Evidently, an effect algebra C with unit w is isomorphic to the horizontal sum $A \dotplus B$ iff there are morphisms $\alpha: A \to C$, $\beta: B \to C$ such that (i) α is an isomorphism of A onto a sub-effect algebra $\alpha(A)$ of C, (ii) β is an isomorphism of B onto a sub-effect algebra $\beta(B)$ of C, (iii) $\alpha(A) \cap \beta(B) = \{0, w\}$, (iv) $\alpha(A) \cup \beta(B) = C$, and (v) if $x \in \alpha(A)$, $y \in \beta(B)$, and $x \oplus y$ is defined, then $x = 0$ or $y = 0$.

If effect algebras and their morphisms are organized into a category, the Cartesian product is the categorical product and the horizontal sum is the categorical coproduct. In this category, the *tensor product* of effect algebras A and B is defined to be an effect algebra $A \otimes B$ together with a bimorphism $\otimes: A \times B \to A \otimes B$ such that (i) $A \otimes B$ is generated by all elements of the form $a \otimes b$ with $a \in A$ and $b \in B$ and (ii) if C is any effect algebra and $\theta: A \times B \to C$ is a bimorphism, there is a morphism $\theta': A \otimes B \to C$ such that $\theta(a, b) = \theta'(a \otimes b)$ for all $a \in A$, $b \in B$ (Bennett and Foulis, 1993). The *interval-algebra tensor product* is defined in the same way, but in the category of interval algebras.

If B is a Boolean algebra, then $A \otimes B$ is the Pták sum of A and B (Foulis and Pták, n.d.). The tensor product $C_n \otimes C_m$ of chains is the chain C_{nm}.

We do not know of an example of effect algebras A and B that fail to have a tensor product. In Dvurečenskij (n.d.) it is shown that $A \otimes B$ exists iff there is a bimorphism with domain $A \times B$. In Section 9 below, we show that any two interval algebras have an interval-algebra tensor product. We do not know whether the interval-algebra tensor product of interval algebras coincides with their tensor product in the larger category of all effect algebras. Related definitions of tensor products can be found in Dvurečenskij and Pulmannová (1994) and Pulmannová (1985). In what follows, we consider only the interval-algebra tensor product.

7. CARTESIAN PRODUCTS

For the remainder of this paper, we assume that A and B are interval algebras with units u and v and that G and H are the universal groups for A and B, respectively.

Note that $A \times B$ is a subset of the Abelian group $G \times H$. We organize $G \times H$ into a partially ordered Abelian group with positive cone $G^+ \times H^+$. Evidently, as an effect algebra,

$$A \times B = (G^+ \times H^+)[(0,0),(u,v)]$$

so $A \times B$ is again an interval algebra. Moreover, we have the following result.

Theorem 7.1. With $G^+ \times H^+$ as the positive cone, $G \times H$ is the universal group with unit (u, v) for $A \times B$.

Proof. Conditions (i)–(iii) in Theorem 3.2 are obviously satisfied. To verify condition (iv), assume that $\phi: A \times B \to K$ is a K-valued measure. The mappings $\alpha: A \to K$ and $\beta: B \to K$ defined by $\alpha(a) := \phi(a, 0)$ and $\beta(b) := \phi(0, b)$ for $a \in A$, $b \in B$ are K-valued measures; hence, they can be extended to group homomorphisms $\alpha^*: G \to K$ and $\beta^*: H \to K$, respectively. Therefore the mapping $\phi^*: G \times H \to K$ defined by $\phi^*(g, h) := \alpha^*(g) + \beta^*(h)$ for $(g, h) \in G \times H$ is a group homomorphism that extends ϕ. ■

Example 7.2. By Theorem 7.1, the rectangular trellis $\mathrm{RT}(n_1, n_2, \ldots, n_r)$ is an interval algebra and its universal group is $\mathbb{Z}^r$ partially ordered by the standard positive cone $(\mathbb{Z}^+)^r$ and with unit $(n_1, n_2, \ldots, n_r)$.

Let $\Omega_A := \{\omega \in \Omega(A \times B) | \omega(0, v) = 0\}$, $\Omega_B := \{\omega \in \Omega(A \times B) | \omega(u, 0) = 0\}$. If $\omega \in \Omega_A$ and $(a, b) \in A \times B$, then $(0, b) \le (0, v)$, so $\omega(0, b) = 0$, and it follows that

$$\omega(a, b) = \omega((a, 0) \oplus (0, b)) = \omega(a, 0) + \omega(0, b) = \omega(a, 0)$$

Likewise, for $\omega \in \Omega_B$, $\omega(a, b) = \omega(0, b)$.

The mapping $\mu \mapsto \mu_A$ from $\Omega(A)$ to $\Omega(A \times B)$ given by $\mu_A(a, b) = \mu(a)$ for all $(a, b) \in A \times B$ is an affine isomorphism of $\Omega(A)$ onto $\Omega_A \subseteq \Delta(A \times B)$. Likewise, the mapping $\nu \mapsto \nu_B$ from $\Omega(B)$ to $\Omega(A \times B)$ given by $\nu_B(a, b) = \nu(b)$ for all $(a, b) \in A \times B$ is an affine isomorphism of $\Omega(B)$ onto $\Omega_B \subseteq \Omega(A \times B)$. Thus, in the sense of the following theorem, $\Omega(A \times B)$ may be regarded as the "convex hull" of $\Omega(A)$ and $\Omega(B)$.

Theorem 7.3. If $\Omega(A), \Omega(B) \ne \emptyset$ and $\omega \in \Omega(A \times B)$, there is a unique $t \in \mathbb{R}^+[0, 1]$ and there are probability weights $\mu \in \Omega(A)$ and $\nu \in \Omega(B)$ such that $\omega = t\mu_A + (1 - t)\nu_B$.

Proof. We may assume that $\omega \notin \Omega_A \cup \Omega_B$, so that $\omega(0, v), \omega(u, 0) \ne 0$. Let $t := \omega(u, 0)$. Since $\omega(0, v) + \omega(u, 0) = \omega(u, v) = 1$, we have $1 - t = \omega(0, v)$. Evidently, $\mu: A \to \mathbb{R}^+[0, 1]$ defined by $\mu(a) := \omega(a, 0)/t$ for all $a \in A$ is a probability measure on A. Likewise, $\nu: B \to \mathbb{R}^+[0, 1]$ defined by $\nu(b) := \omega(0, b)/(1 - t)$ for all $b \in B$ is a probability measure on B.

Furthermore,

$$\begin{aligned}(t\mu_A + (1-t)\nu_B)(a, b) &= t\mu(a) + (1-t)\nu(b)\\ &= \omega(a, 0) + \omega(0, b)\\ &= \omega(a, b)\end{aligned}$$

Conversely, if $\omega = t\mu_A + (1-t)\nu_B$, then $\omega(u, 0) = t\mu(u) = t$, so t is uniquely determined. ■

Evidently, if $\Omega(B) = \varnothing$, then $\Omega(A \times B) = \Omega_A$ and, if $\Omega(A) = \varnothing$, then $\Omega(A \times B) = \Omega_B$. Of course, $\Omega(A \times B) = \varnothing$ iff $\Omega_A = \Omega_B = \varnothing$.

Corollary 7.4. $\partial_e \Omega(A \times B) = \partial_e \Omega_A \,\dot{\cup}\, \partial_e \Omega_B$.

Example 7.5. Because a chain C_n admits only one probability measure, Corollary 7.4 implies that the space of probability measures on $\mathrm{RT}(n_1, n_2, \ldots, n_r)$ has exactly r extreme points.

8. HORIZONTAL SUMS

The universal group of a horizontal sum of interval algebras is constructed from the quotient group of a direct product, and the following observation on ordering quotient groups will be used in that construction. If U is a subgroup of the partially ordered Abelian group G and $U \cap G^+ = \{0\}$, then G/U can be organized into a partially ordered Abelian group with $(G/U)^+ = G^+/U$. Indeed, G^+/U is closed under addition, and if $g_1 + U = -g_2 + U$ with $g_1, g_2 \in G^+$, then $g_1 + g_2 \in U \cap G^+ = \{0\}$; thus $g_1 = -g_2$, so that $G^+/U \cap -(G^+/U)$ is the zero element of G/H.

Let U be the cyclic subgroup of $G \times H$ generated by $(u, -v)$, let $Q = (G \times H)/U$, and let $\eta\colon G \times X \to Q$ be the canonical epimorphism. Thus, for $n \in \mathbb{Z}$, $g \in G$, $h \in H$, we have

$$\eta(g, h) = \eta(g + nu, h - nv)$$

Because $U \cap (G^+ \times H^+) = \{(0, 0)\}$, it follows that Q can be organized into a partially ordered group with $Q^+ := \eta(G^+ \times H^+)$ as a generating positive cone. Evidently $w := \eta(u, 0) = \eta(0, v)$ is a nonzero element of Q^+, every element in Q^+ is a sum of a sequence of elements in the interval $Q^+[0, w]$, and Q^+ generates Q.

Define $\alpha\colon A \to Q^+[0, w]$ and $\beta\colon B \to Q^+[0, w]$ by $\alpha(a) = \eta(a, 0)$ and $\beta(b) = \eta(0, b)$ for all $a \in A$, $b \in B$. Thus, α and β are effect-algebra isomorphisms of A and B onto sub-effect algebras $\alpha(A)$ and $\beta(B)$, respectively, of $Q^+[0, w]$.

Theorem 8.1. With the notation above, $Q^+[0, w] = \alpha(A) \dot{+} \beta(B)$ and, with w as the unit, Q is the universal group for the horizontal sum $\alpha(A) \dot{+} \beta(B)$.

Proof. Suppose $\eta(g, h) \in \alpha(A) \cap \beta(B)$. Then there are elements $a \in A$, $b \in B$, and $m, n \in \mathbb{Z}$ such that $a = g + nu$, $h = nv$, $g = -mu$, and $b = h - mv$. Therefore, $a = (n - m)u$ and $b = (n - m)v$. Since $0 \leq a \leq u$, it follows that $n - m = 0$ or $n - m = 1$, and so $\eta(g, h) = 0$ or $\eta(g, h) = w$. Therefore, $\alpha(A) \cap \beta(B) = \{0, w\}$.

Suppose $q \in Q^+[0, w]$. Since $q \in Q^+$, there exists $g \in G^+$ and $h \in H^+$ with $q = \eta(g, h)$. Since $w - \eta(g, h) \in Q^+$, there is an integer n such that $0 \leq g \leq (n + 1)u$ in G and $0 \leq h \leq -nv$ in H. Therefore, $n = 0$ and $q = \eta(g, h) \in \alpha(A)$ or else $n = -1$ and $q = \eta(g, h) \in \beta(b)$. Consequently, $Q^+[0, w] = \alpha(A) \cup \beta(B)$.

Suppose $a \in A$, $b \in B$, and $\alpha(a) + \beta(b) \in Q^+[0, w]$, that is, $w - \eta(a, b) \in Q^+$. Then there is an integer n such that $0 \leq a \leq (n + 1)u$ in G and $0 \leq b \leq -nv$ in H, and it follows that $n \in \{0, -1\}$, so that $b = 0$ or $a = 0$. Therefore, $Q^+[0, w]$ is the horizontal sum of its sub-effect algebras $\alpha(A)$ and $\beta(B)$.

Let $\phi: Q^+[0, w] \to K$ be a K-valued measure. To complete the proof, we only have to show that ϕ can be extended to a group homomorphism $\phi^*: Q \to K$. The K-valued measures $\phi \circ \alpha: A \to K$ and $\phi \circ \beta: B \to K$ can be extended to group homomorphisms $(\phi \circ \alpha)^*: G \to K$ and $(\phi \circ \beta)^*: H \to K$. The group homomorphism $\xi: G \times H \to K$ defined by $\xi(g, h) = (\phi \circ \alpha)^*(g) + (\phi \circ \beta)^*(h)$ satisfies the condition $\xi(u, -v) = 0$, so there exists a group homomorphism $\phi^*: Q \to K$ such that $\phi^* \circ \eta = \xi$. For $a \in A$,

$$\phi^*(\alpha(a)) = \phi^*(\eta(a, 0)) = \xi(a, 0) = (\phi \circ \alpha)^*(a) = \phi(\alpha(a))$$

and likewise, $\phi^*(\beta(b)) = \phi(\beta(b))$. Because $Q^+[0, w] = \alpha(A) \cup \beta(B)$, it follows that ϕ^* is an extension of ϕ. ■

We omit the straightforward proof of the following theorem.

Theorem 8.2. If $\Omega(A), \Omega(B) \neq \emptyset$, the mapping $\Phi: \Omega(A) \times \Omega(B) \to \Omega(A \dot{+} B)$ given by $\Phi(\mu, \nu)(x) = \mu(x)$ for $x \in A$ and $\Phi(\mu, \nu)(x) = \nu(x)$ for $x \in B$ is an affine isomorphism of $\Omega(A) \times \Omega(B)$ onto $\Omega(A \dot{+} B)$ and Φ maps $\partial_e \Omega(A) \times \partial_e \Omega(B)$ onto $\partial_e \Omega(A \cup B)$.

As a consequence of Theorem 8.2, a horizontal sum of polychains of height three or more admits only one probability measure.

In the next theorem, we illustrate the use of Theorem 8.1 by computing the universal group of the horizontal sum $C_n \dot{+} C_m$ of two chains. We denote the additive group of integers modulo d by $\mathbb{Z}_d$, with the understanding that $\mathbb{Z}_1 = \{0\}$, and we denote the canonical epimorphism $\mathbb{Z} \to \mathbb{Z}_d$ by δ.

Theorem 8.3. Let n, m be positive integers, let d be the greatest common divisor of n and m and let h and k be integers such that $hn + km = d$. Let $J := \mathbb{Z} \times \mathbb{Z}_d$ and $J^+ := \{(mx/d + ny/d, \delta(hx - ky)) | (x, y) \in \mathbb{Z}^+ \times \mathbb{Z}^+\}$. Then J is partially ordered by the cone J^+ and, with $w := (nm/d, \delta(0)) \in J^+$, J is the universal group for $J^+[0, w]$. Furthermore, the mappings $\alpha: C_n \to J^+[0, w]$ and $\beta: C_m \to J^+[0, w]$ given by $\alpha(x) := (mx/d, \delta(hx))$ and $\beta(y) := (ny/d, \delta(-ky))$ for $x \in C_n = \mathbb{Z}^+[0, n]$ and $y \in C_m = \mathbb{Z}^+[0, m]$ are effect-algebra isomorphisms of C_n and C_m onto sub-effect algebras $\alpha(C_n)$ and $\beta(C_m)$ of $J^+[0, w]$. Also, $J^+[0, w] = \alpha(C_n) \dot{+} \beta(C_m)$.

Proof. We sketch the proof, leaving the details to the interested reader. The mapping $\Psi: \mathbb{Z} \times \mathbb{Z} \to J$ defined for $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ by $\Psi(x, y) := (mx/d + ny/d, \delta(hx - ky))$ is a group epimorphism and $\ker(\Psi)$ is the cyclic subgroup U of $\mathbb{Z} \times \mathbb{Z}$ generated by $(n, -m)$. Therefore, if $\eta: \mathbb{Z} \times \mathbb{Z} \to Q := (\mathbb{Z} \times \mathbb{Z})/U$ is the canonical epimorphism, there is a group isomorphism $\xi: Q \to J$ such that $\xi(\eta(x, y)) = \Psi(x, y)$. Using the isomorphism ξ, we obtain the present theorem directly from Theorem 8.1. ■

As a corollary of Theorem 8.3, we note that if n and m are relatively prime, then the universal group of $C_n \dot{+} C_m$ is isomorphic to $J := \mathbb{Z}$ with the nonstandard cone $J^+ := \{mx + ny | x, y \in \mathbb{Z}^+\}$ and with the unit $w := nm$. Under this isomorphism, C_n corresponds to $\{mx | x \in C_n\}$ and C_m corresponds to $\{ny | y \in C_m\}$.

Example 8.4. Using Theorem 8.1 and mathematical induction on n, it can be shown that $MO(n)$ in Example 6.3 can be realized as $G^+[0, u]$, where $G = \mathbb{Z}^{n+1}$, $u := (1, 1, 1, \ldots, 1)$, the $2n + 2$ elements of $MO(n)$ are $0, u$, the n elements $a_1 := (1, 0, 0, 0, \ldots, 0)$, $a_2 := (1, 1, 0, 0, \ldots, 0)$, $a_3 := (1, 1, 1, 0, \ldots, 0), \ldots, a_n := (1, 1, 1, 1, \ldots, 1, 0)$, and n more elements of the form $b_i := u - a_i$ for $i = 1, 2, \ldots, n$. Here G^+ is the subcone of the standard positive cone $(\mathbb{Z}^+)^{n+1}$ consisting of all nonnegative-integer linear combinations of $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots$, and b_n.

The interval algebra $MO(n)$ in Example 8.4 is the quantum logic affiliated with measurements of the spin component in n different directions of a spin-1/2 particle, and $MO(n) \otimes MO(n)$ is the quantum logic for the anticorrelated spin experiments used to test the Bell inequalities (Kläy, 1988). Using the result of Example 8.4 and Theorem 9.1 in the next section, we can compute the universal group of $MO(n) \otimes MO(n)$.

9. TENSOR PRODUCTS

The tensor product $G \otimes H$ of the Abelian groups G and H can be organized into a partially ordered Abelian group with positive cone $(G \otimes H)^+$ consisting of all sums of finite sequences of pure tensors of the form $g \otimes h$ with $g \in G^+$ and $h \in H^+$ (Goodearl and Handleman; 1986, Proposition 2.1).

Theorem 9.1. With $(G \otimes H)^+$ defined as above:

(i) If $\mu \in \Omega(A)$ and $\nu \in \Omega(B)$, there is a group homomorphism $\sigma: G \otimes H \to \mathbb{R}$ such that $\sigma(a \otimes b) = \mu(a)\nu(b)$ for all $a \in A = G^+[0, u]$, $b \in B = H^+[0, v]$.

(ii) $0 \neq u \otimes v \in (G \otimes H)^+$.

(iii) $(H \otimes H)^+$ is a generating cone in $G \otimes H$.

(iv) Every element in $(G \otimes H)^+$ is the sum of a finite sequence of pure tensors $a \otimes b \in (G \otimes H)^+[0, u \otimes v]$ for $a \in A$, $b \in B$.

(v) As an effect algebra, $(G \otimes H)^+[0, u \otimes v]$ is generated by all pure tensors of the form $a \otimes b$, with $a \in A$, $b \in B$.

(vi) If K is an Abelian group and $\theta: A \times B \to K$ is a K-valued bimeasure, there exists a group homomorphism $\theta^*: G \otimes H \to K$ such that $\theta^*(a \otimes b) = \theta(a, b)$ for all $a \in A$, $b \in B$.

(vii) With the mapping $A \times B \to (G \otimes H)^+(0, u \otimes v)$ given by $(a, b) \mapsto a \otimes b$ as the canonical bimorphism, $(G \otimes H)^+(0, u \otimes v)$ is the interval-algebra tensor product of the interval algebras A and B.

(viii) $G \otimes H$ is the universal group for the interval algebra $(G \otimes H)^+[0, u \otimes v]$.

Proof. (i) Let $\mu \in \Omega(A)$, $\nu \in \Omega(B)$. Because G and H are the universal groups for A and B, we can extend μ and ν to group homomorphisms $\mu^*: G \to \mathbb{R}$ and $\nu^*: H \to \mathbb{R}$. Since the mapping $G \times H \to \mathbb{R}$ given by $(g, h) \mapsto \mu^*(g)\nu^*(h)$ is a group bihomomorphism, there exists a group homomorphism $\sigma: G \otimes H \to \mathbb{R}$ such that $\sigma(g \otimes h) = \mu^*(g)\nu^*(h)$ for all $g \in G$, $h \in H$.

(ii) By Bennett and Foulis (n.d.), Theorem 6.7, there are probability measures $\mu \in \Omega(A)$, $\nu \in \Omega(B)$ with $\mu(u) = \nu(v) = 1$. Let σ be the corresponding group homomorphism as in (i). Then $\sigma(u \otimes v) = \mu(u)\nu(v) = 1$, so $u \otimes v \neq 0$.

(iii) If $g \in G$, and $h \in H$, we can write $g = g_1 - g_2$ and $h = h_1 - h_2$, with $g_1, g_2 \in G^+$ and $h_1, h_2 \in H^+$, and it follows that

$$g \otimes h = (g_1 \otimes h_1 + g_2 \otimes h_2) - (g_1 \otimes h_2 + g_2 \otimes h_1) \in (G \otimes H)^+ - (G \otimes H)^+$$

Since every element in $G \otimes H$ is a sum of pure tensors $g \otimes h$, it follows that $(G \otimes H)^+$ is a generating cone for $G \otimes H$.

(iv) If $a \in A$, $b \in B$, then $0 \leq a \leq u$ in B and $0 \leq b \leq v$ in H, so $0 \leq a \otimes b \leq u \otimes v$ in $G \otimes H$. If $g \in G^+$ and $h \in H^+$, then $g = \sum_i a_i$ and $h = \sum_j b_j$ for $a_i \in A$ and $b_j \in B$, and it follows that $g \otimes h = \sum_i \sum_j a_i \otimes b_j$ with $a_i \otimes b_j \in (G \otimes H)^+[0, u \otimes v]$.

(v) If $t \in (G \otimes H)^+[0, u \otimes v]$, then $t = \sum_i a_i \otimes b_i$ with $a_i \in A$, $b_i \in B$ by (iv) and, since $t \le u \otimes v$, we have $t = \bigoplus_i a_i \otimes b_i$ in the interval effect algebra $(G \otimes H)^+[0, u \otimes v]$.

(vi) See Foulis and Bennett (1994), Theorem 9.4.

(vii) By (v), $(G \otimes H)^+[0, u \otimes v]$ is generated as an effect algebra by all $a \otimes b$, $a \in A$, $b \in B$. Suppose that C is an interval algebra with unit w and universal group K, and let $\theta: A \times B \to C$ be a bimorphism. Then $\theta: A \times B \to K$ is a bimeasure, so it induces a group homomorphism $\theta^*: G \otimes H \to K$ as in (vi). We have $\theta^*(u \otimes v) = \theta(u, v) = w$ and, by (iv), θ^* maps $(G \otimes H)^+$ into K^+, so the restriction of θ^* to the interval $(G \otimes H)^+[0, u \otimes v]$ provides a morphism $\theta': (G \otimes H)^+[0, u \otimes v] \to K^+[0, w] = C$ such that $\theta(a, b) = \theta'(a \otimes b)$ for all $a \in A$, $b \in B$.

(viii) Let $\phi: (G \otimes H)^+[0, u \otimes v] \to K$ be a K-valued measure. The mapping $\theta: A \times B \to K$ given by $\theta(a, b) := \phi(a \otimes b)$ for $a \in A$, $b \in B$ is a K-valued bimeasure, so, by (vi), there exists a group homomorphism $\phi^*: G \otimes H \to K$ such that $\phi^*(a \otimes b) = \theta(a, b) = \phi(a \otimes b)$ for all $a \in A$, $b \in B$, and it follows from (v) that ϕ^* is an extension of ϕ. Therefore, by Theorem 3.2, $G \otimes H$ is the universal group for $(G \otimes H)^+[0, u \otimes v]$. ■

10. χ-ALGEBRAS

Cartesian products, horizontal sums, and tensor products have perspicuous interpretations in quantum logic (Foulis, 1989). For instance, M. Kläy (1988) has made effective use of $MO(2) \otimes MO(2)$ to study the Bohm version of the EPR Gedankenexperiment. This suggests that the following problem warrants consideration:

The CHT Problem. Given a class $\mathscr{C}$ of effect algebras, characterize the class CHT($\mathscr{C}$) consisting of the effect algebras in $\mathscr{C}$ and all effect algebras that can be obtained from these algebras by iteratively forming finite Cartesian products (C), horizontal sums (H), and tensor products (T).

For a more general CHT problem, the word "finite" may be omitted.

If only Cartesian products and horizontal sums are allowed, the corresponding "CH" problem was solved for $\mathscr{C} = \{\mathbf{2}\}$ by Dacey (1968).

Dacey's CH(2) Theorem. An interval algebra A can be obtained starting with copies of $\mathbf{2}$ and iteratively forming finite Cartesian products and horizontal sums iff A is a finite orthomodular lattice and there do not exist four distinct atoms a, b, c, and d in A such that $a \oplus b$, $b \oplus c$, and $c \oplus d$ are defined and $a \oplus c$, $b \oplus d$, and $a \oplus d$ are not defined.

In this section, we make a modest start on the problem of characterizing CHT($\mathbf{2}$) by singling out a class of finite interval algebras that contains

2 and is closed under the formation of finite Cartesian products, horizontal sums, and interval-algebra tensor products.

If X is a nonempty set and $M \subseteq X$, the *characteristic set function* $\chi_M : X \to \{0, 1\} \subseteq \mathbb{Z}$ is defined as usual by $\chi_M(x) := 1$ if $x \in M$ and $\chi_M(x) := 0$ if $x \in X \setminus M$. If the group $\mathbb{Z}^X$ is partially ordered by the standard positive cone $(\mathbb{Z}^+)^X$, then the interval algebra $(\mathbb{Z}^+)^X[0, \chi_X]$ consists of all the characteristic set functions χ_M for $M \subseteq X$; hence, as a poset, it is isomorphic to the power set of X. We refer to χ_X as the standard unit in the group $\mathbb{Z}^X$.

Definition 10.1. An interval algebra A is called a *χ-algebra over X* iff the universal group of A is $G := \mathbb{Z}^X$, the positive cone of G is contained in $(\mathbb{Z}^+)^X$, and the unit is $u := \chi_X$.

By the following lemma, every element of a χ-algebra over X is a characteristic set function for a subset of X, so χ-algebras are closely related to the concrete logics of Pták and Pulmannová (1991, p. 2).

Lemma 10.2. Let X be a nonempty set, let $G := \mathbb{Z}^X$ be partially ordered by a positive cone G^+, let $u = \chi_X \in G$ be the standard unit, and suppose that G is the universal group with unit u for $G^+[0, u]$. Then $G^+ \subseteq (\mathbb{Z}^+)^X$ iff $G^+[0, u]$ consists only of characteristic set functions.

Proof. If $G^+ \subseteq (\mathbb{Z}^+)^X$, $g \in G^+[0, u]$, and $x \in X$, then $g(x) \in \mathbb{Z}^+$ and $u(x) - g(x) = 1 - g(x) \in \mathbb{Z}^+$, so $g(x)$ is either zero or one. Conversely, if every function $g \in G^+[0, u]$ takes on only the values zero and one, then $G^+[0, u] \subseteq (\mathbb{Z}^+)^X$ and, since every element in G^+ is a sum of a finite sequence of elements of $G^+[0, u]$, it follows that $G^+ \subseteq (\mathbb{Z}^+)^X$. ■

If we say that A is a χ-algebra, we mean that it is (or is isomorphic to) a χ-algebra over some nonempty set X. As an obvious consequence of Lemma 10.2, a χ-algebra cannot contain any isotropic elements, and therefore *every χ-algebra is an orthoalgebra.*

By Example 4.3, $\mathbf{2}^r$ is a χ-algebra over $X := \{1, 2, 3, \ldots, r\}$ and, by Example 8.4, $MO(n)$ is a χ-algebra over $X := \{1, 2, \ldots, n+1\}$. Since the universal group of a χ-algebra must be torsion-free, the polychains of height three or more and the diamond in Example 4.7 give examples of interval algebras that are not χ-algebras.

Lemma 10.3. If A and B are χ-algebras, so are $A \times B$ and $A \dotplus B$.

Proof. Let $A = G^+[0, u]$, $B = H^+[0, v]$, $G = \mathbb{Z}^X$, $H = \mathbb{Z}^Y$, $G^+ \subseteq (Z^+)^X$, and $H^+ \subseteq (\mathbb{Z}^+)^Y$, with standard units u and v, and suppose that G, H are the universal groups for A, B, respectively. Without loss of generality, we may assume that $X \cap Y = \emptyset$, so that, in what follows, we can make the canonical identification of $G \times H = \mathbb{Z}^X \times \mathbb{Z}^Y$ with $\mathbb{Z}^{X \cup Y}$ by regarding an ordered pair $(g, h) \in G \times H$ as the function on $X \cup Y$ defined by $(g, h)(x) = g(x)$ for $x \in X$ and $(g, h)(y) = h(y)$ for $y \in Y$. Note that (u, v) is then the standard unit in $\mathbb{Z}^{X \cup Y}$ and $(g, h) \in (\mathbb{Z}^+)^{X \cup Y}$ iff $g \in (\mathbb{Z}^+)^X$ and $h \in (\mathbb{Z}^+)^Y$. Therefore, $A \times B$ is a χ-algebra by Theorem 7.1.

Let U be the cyclic subgroup of $G \times H$ generated by $(u, -v)$ and let $\eta: \mathbb{Z}^{X \cup Y} \to Q := \mathbb{Z}^{X \cup Y}/U$ be the canonical epimorphism. Choose and fix $a \in X$, $b \in Y$, let $Z := X \cup Y \backslash \{b\}$, $R := \mathbb{Z}^Z$, and define the epimorphism $\Phi: \mathbb{Z}^{X \cup Y} \to R$ by

$$\Phi(g, h)(z) := \begin{cases} g(z) + h(b) & \text{if } z \in X \\ g(a) + h(z) & \text{if } z \in Y \backslash \{b\} \end{cases} \qquad \text{for all } z \in Z$$

Since $\ker(\Phi) = U$, there is an isomorphism $\phi: Q \to R$ such that $\phi \circ \eta = \Phi$. Under this isomorphism, we identify the universal group Q in Theorem 8.1 with R, noting that the unit in R is the standard unit. If either g is a characteristic set function and $h = 0$ or $g = 0$ and h is a characteristic set function, then $\Phi(g, h)$ is a characteristic set function; hence, by Lemma 10.2, $R^+ := \phi(Q^+) \subseteq (\mathbb{Z}^+)^Z$, and it follows that $A \dotplus B$ is a χ-algebra. ∎

If A is a χ-algebra over a finite set $X = \{x_1, x_2, \ldots, x_n\}$, it is clear from Lemma 10.2 that A can contain at most 2^n elements. Conversely, if A is a *finite* χ-algebra over X, then the group $\mathbb{Z}^X$ has a finite set of generators, namely A; hence, it has finite rank, so X is a finite set.

Lemma 10.4. If A and B are finite χ-algebras, then so is the interval-algebra tensor product $A \otimes B$.

Proof. We use the same notation as in the proof of Lemma 10.3, but assume that X and Y are finite sets. Then there is a canonical isomorphism $\psi: G \otimes H \to S := \mathbb{Z}^{X \times Y}$ such that, for $g \in G$, $h \in H$, $x \in X$ and $y \in Y$, we have $\psi(g \otimes h)(x, y) = g(x)h(y) \in \mathbb{Z}$. Note that $\psi(u \otimes v)$ is the standard unit in S and that, if g and h are characteristic set functions, so is $\psi(g, h)$. Therefore, by Theorem 9.1 and Lemma 10.2, $S^+[0, \psi(u \otimes v)]$ is effective as the universal group for an isomorphic copy of $A \otimes B$ and S^+ is contained in the standard positive cone $(\mathbb{Z}^+)^{X \times Y}$. ∎

Example 10.5. Let $G := \mathbb{Z}^4$ be partially ordered by the cone $G^+ := \{(x, y, z, w) \in (\mathbb{Z}^+)^4 | x + y + z \geq w\}$. Then the χ-algebra $A := G^+[(0, 0, 0, 0), (1, 1, 1, 1)]$, which contains exactly 14 elements, is the smallest *proper* orthoalgebra, i.e., the smallest orthoalgebra that is not an orthomodular poset. It is the logic of the *Wright triangle* (Foulis *et al.*, 1992, Example 2.13) and it does not belong to CHT(**2**). There are exactly five elements of $\partial_e(\Omega(A))$, four of which are given by the restrictions to A

of the projection mappings $\pi_x, \pi_y, \pi_z, \pi_w$ on $\mathbb{Z}^4$. The fifth is the restriction to A of $\frac{1}{2}(\pi_x + \pi_y + \pi_z - \pi_w)$ corresponding to the condition $x + y + z \geq w$ that determines G^+.

By Lemmas 10.3 and 10.4, every effect algebra in CHT(**2**) is a finite χ-algebra; hence, it is an orthoalgebra. However, by Example 10.5, there are finite χ-algebras that are not in CHT(**2**), so the problem of characterizing CHT(**2**) remains open.

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